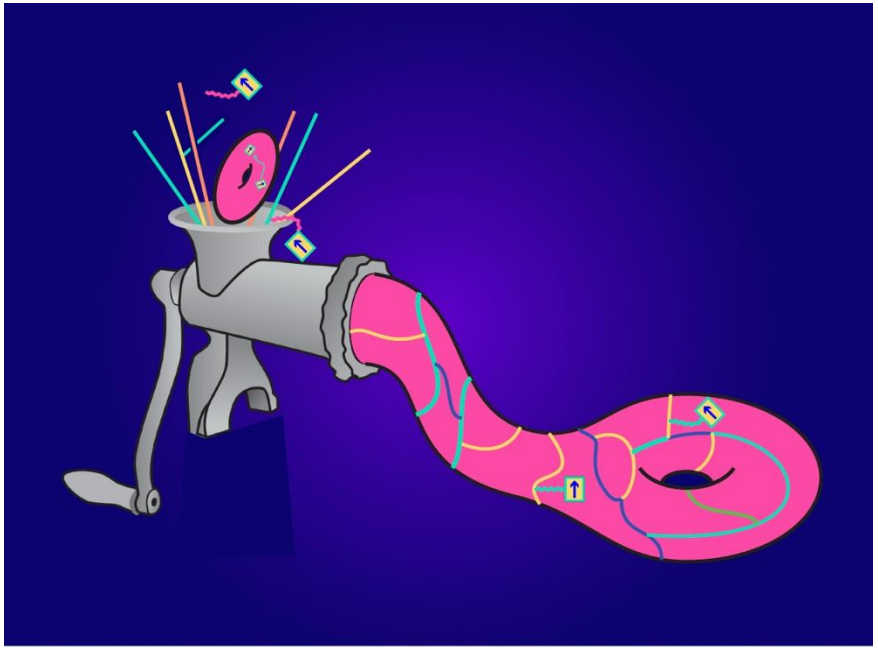
Topological Defects and Non-Invertible Symmetries

Paul Fendley

All Souls College, Oxford



Mainly based on work with David Aasen and Roger Mong (2016,2020) and Luisa Eck (2023,2024)

Canonical example is Kramers-Wannier duality

AUGUST 1, 1941 PHYSICAL REVIEW VOLUME 60

Statistics of the Two-Dimensional Ferromagnet. Part I

H. A. KRAMERS, *University of Leiden, Leiden, Holland*

AND

G. H. WANNIER, *University of Texas, Austin, Texas*[1]

(Received April 7, 1941)

In an effort to make statistical methods available for the treatment of cooperational phenomena, the Ising model of ferromagnetism is treated by rigorous Boltzmann statistics. A method is developed which yields the partition function as the largest eigenvalue of some finite matrix, as

It is shown that these matrices possess a symmetry property which permits location of the Curie temperature if it exists and is unique. It lies at

$$J/kT_c = 0.8814$$

K-W showed that the energy spectra of the ordered and disordered phases of the Ising quantum spin chain are identical, i.e. found a "duality".

But there are two ground states in ordered phase, and one in the disordered.

Map cannot be invertible!

We now understand how to generalize

The key is to write lattice models in terms of a fusion category.

One then can construct topological defects that implement such maps.

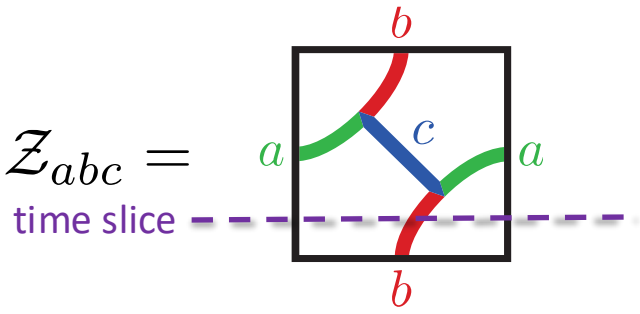
Yields exact results without integrability, significantly generalizing Kramers-Wannier duality.

Much recent work has been done on such “symmetries” in QFT, but I will stick to lattice.

Topological defects generate categorical/topological/higher/non-invertible "symmetries" and "dualities"

The partition function in the presence of topological defects is independent of local deformations of the defects, both on the lattice and in the continuum.

Just need to keep track of how they branch/fuse, and how they wrap around cycles. Schematically:



With defects in spacetime, can still work in a Hilbert-space formalism. The defect-line creation operators necessarily commute with the Hamiltonian/transfer matrix.

Symmetries and dualities are in quotes because these operators are in general non-unitary. Sometimes they are not even invertible.

Finding lattice topological defects

People found the duality defects in Ising by brute force, and that's not easy.

Grimm-Schutz 93; Oshikawa-Affleck 98

Beyond a few other simple examples, it's very difficult to get much further.

Our work exploits the fact that many interesting statistical mechanical models are conveniently described in terms of graphs and the corresponding knot and link invariants.

Temperley-Lieb 71; Fortuin-Kasteleyn 71; Baxter-Kelland-Wu 76; Jones '80s

The more systematic version is to define these models using fusion categories. The payoff is that exact topological defects follow automatically.

Feiguin et al 2006; Aasen-Fendley-Mong 2020

There is such a thing as a free lunch!

Outline for today

(and probably beginning of tomorrow)

Statistical mechanics from fusion categories

1) What is a fusion category and what does it have to do with knot polynomials?

2) Topological defects occur automatically in lattice models defined using the category

3) Their properties (e.g. what happens when you glue them together) follow directly

Outline for rest of tomorrow

The uses of topological defects

Use #1: Generalized Kramers-Wannier duality

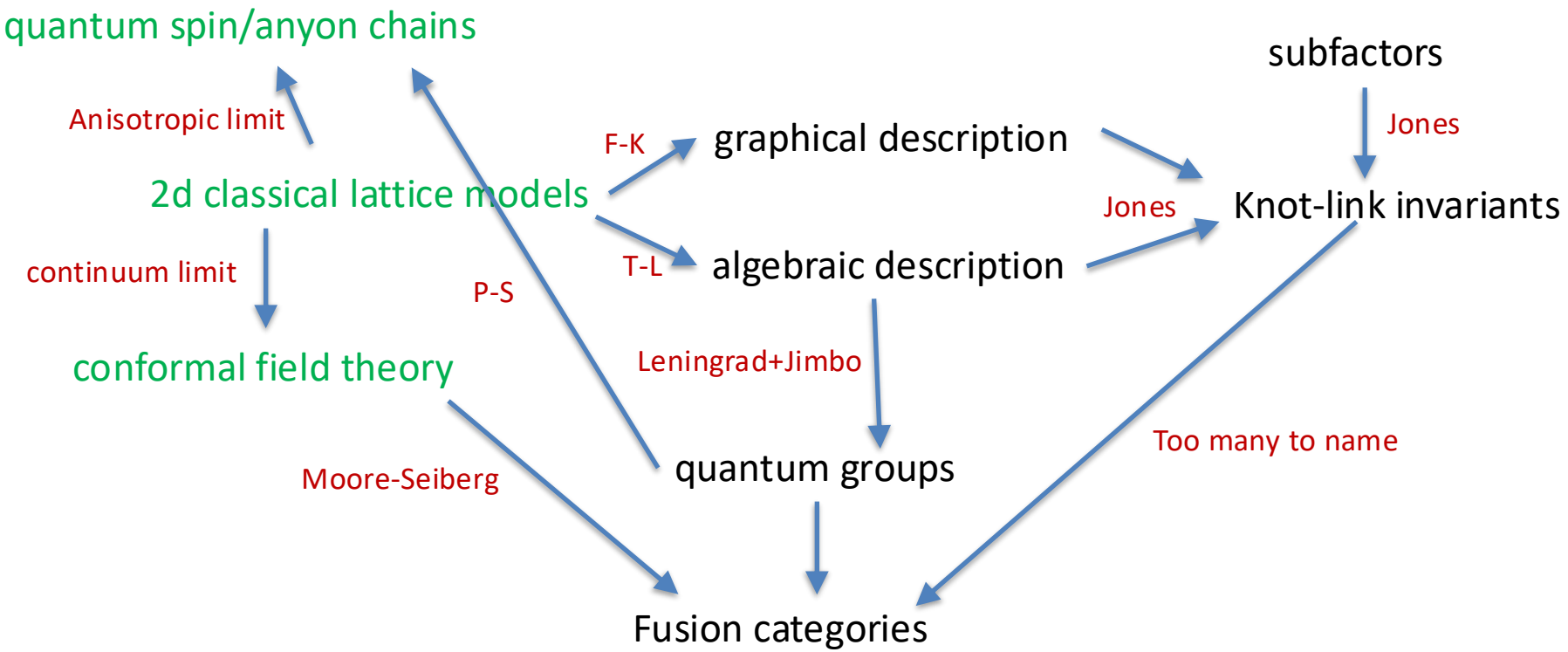
Use #2: Exact degeneracies for ground states and for kinks

Use #3: Non-obvious non-invertible mappings
(from XXZ to the Rydberg-blockade ladder)

Use #4: Exact lattice computation of g factor ratios and conformal spins

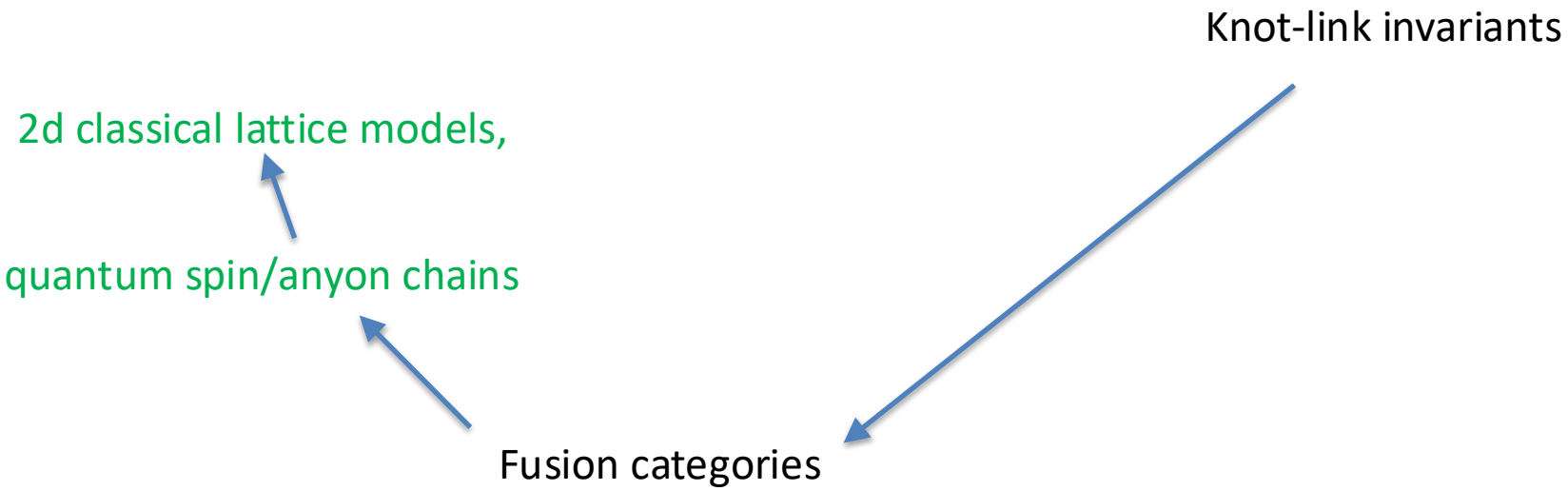
How the mathematics relates to the physics

according to a statistical mechanic



Almost all the mathematical background was understood in the 80s…

Today

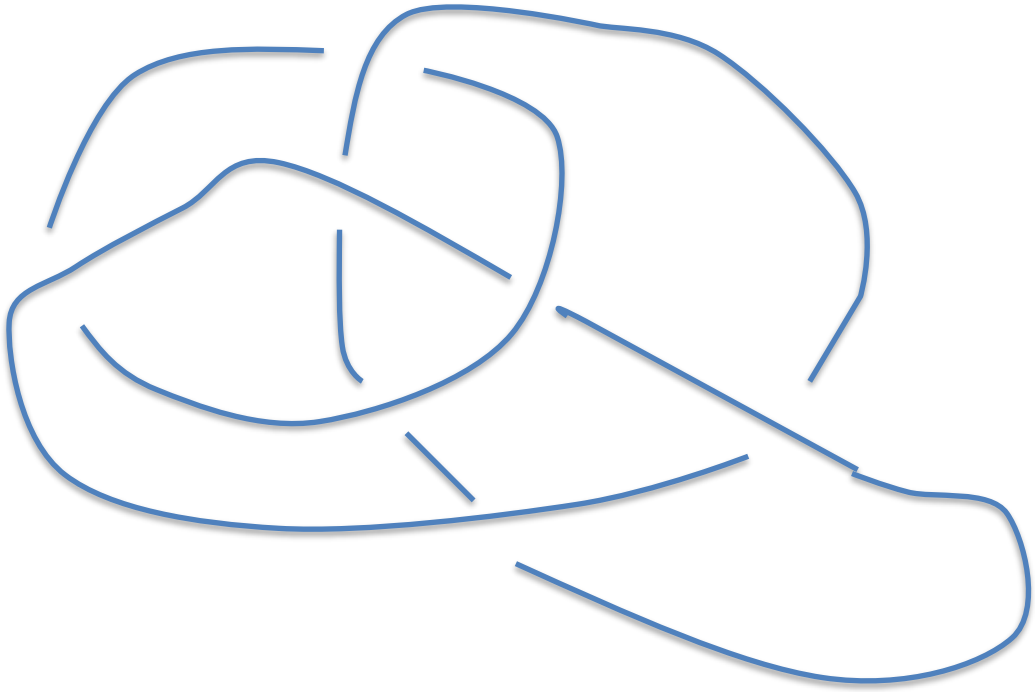


Ask not what I can do for fusion categories, ask what fusion categories can do for me.

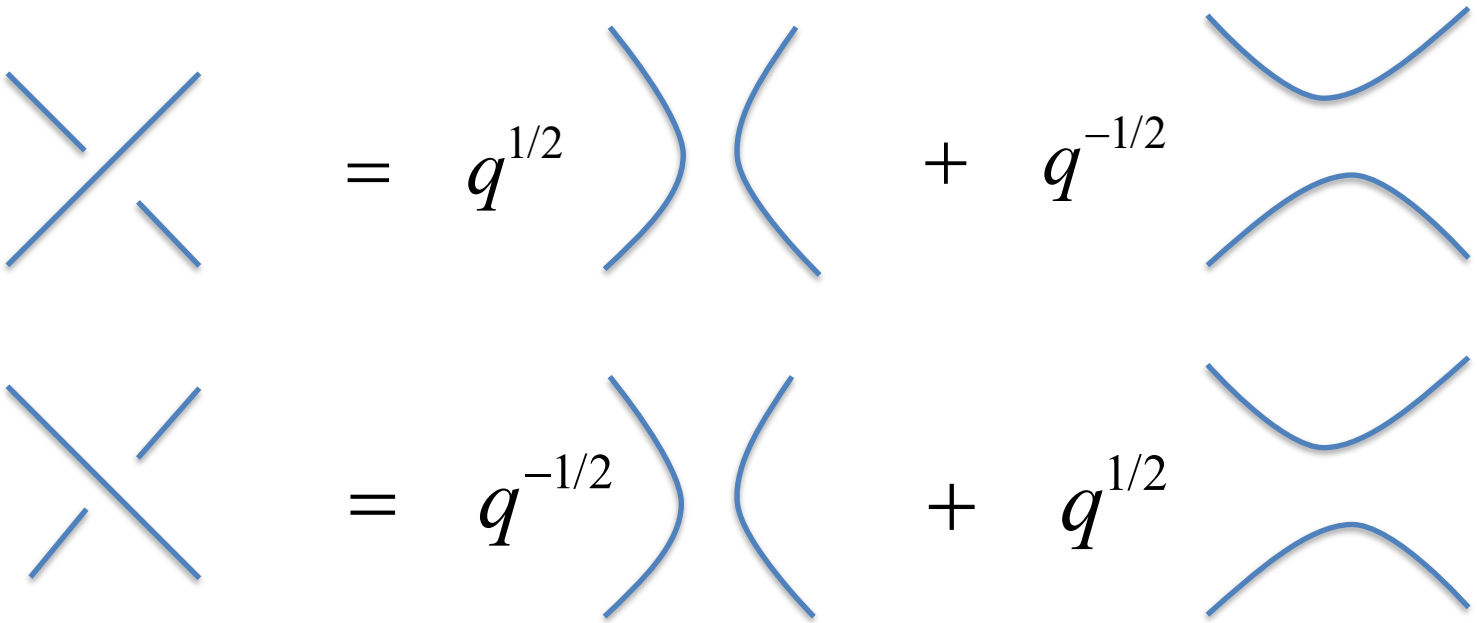
Knot and link invariants

A knot or link invariant such as the Jones polynomial depends only on the topology of the knot.

The first step is to project the knot/link onto the plane:

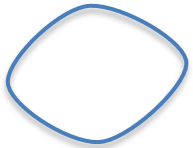


Then ``resolve'' each over/undercrossing and turn each knot/link into a sum over planar graphs. For the Jones polynomial:



This turns each link into a sum over graphs of closed loops.

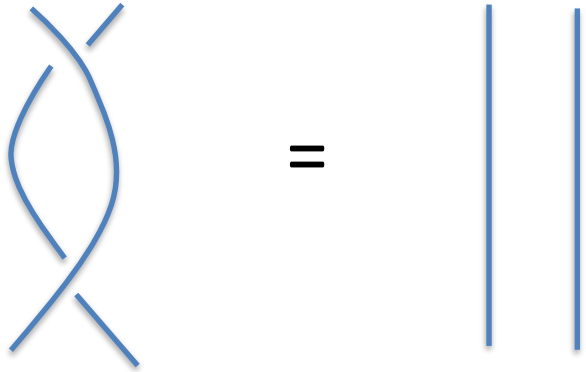
To evaluate the Jones polynomial (in q), replace each loop with



$$= -q - q^{-1} = d$$

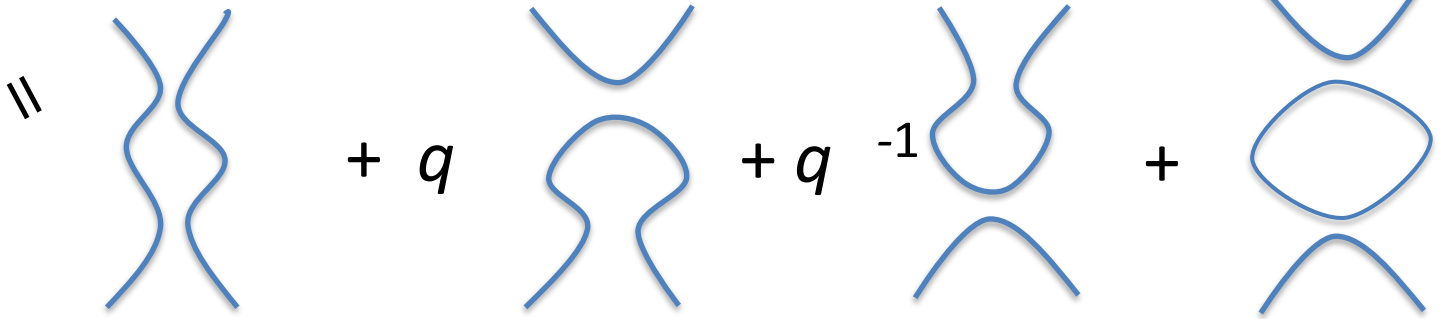
To be topologically invariant and independent of how you project onto the plane, must satisfy the Reidemeister moves:

#2

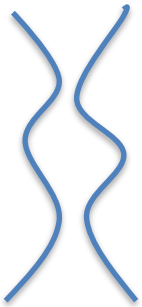


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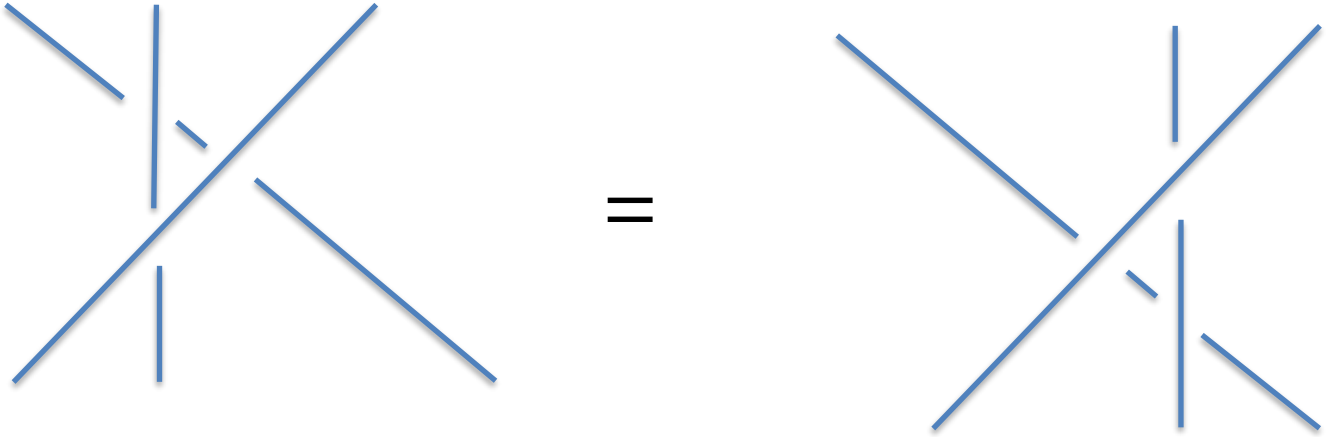
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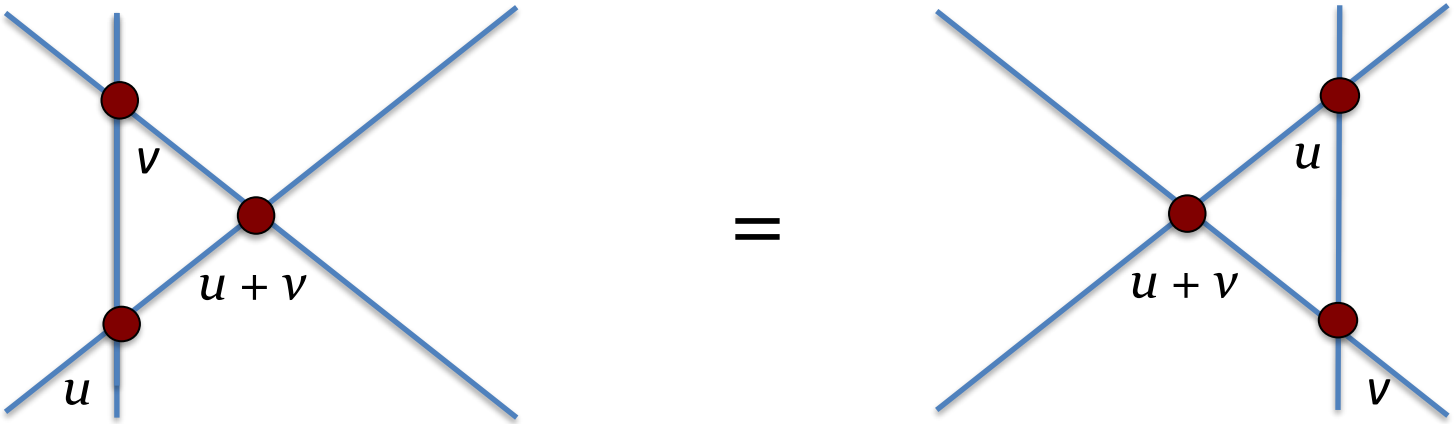
from getting rid of closed loop and multiplying term by

$$d = -q - q^{-1}$$

#3:



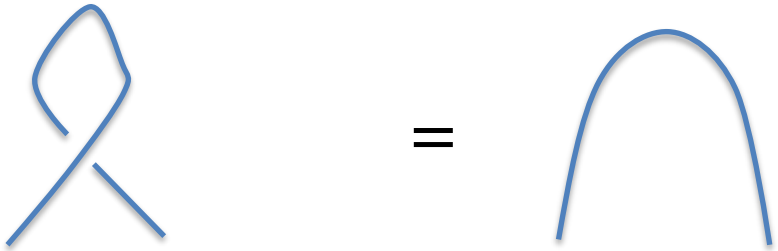
Solutions can be found by taking a limit of the Yang-Baxter equation!



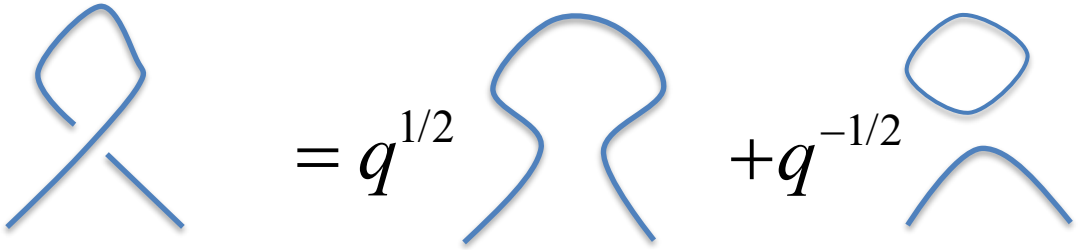


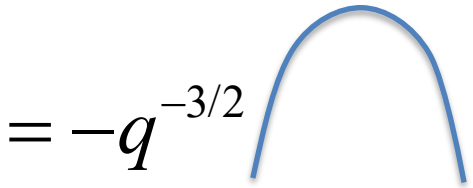
One extremely important subtlety: Need

#1:



But instead:





To have a topological invariant, make each link a ribbon, and keep track of twists.
Then multiply by $q^{3w/2}$, where w = #(signed twists) = writhe.

In physics: conformal spin; anyonic spin

Fusion categories allow us to generalize and connect to lattice statistical mechanics

Even in Jones's original paper, the connection was there.

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

BY VAUGHAN F. R. JONES[2]

For real t, D. Evans pointed out that an explicit representation of A_n on $\mathbf{C}^{2n+2}$ was discovered by H. Temperley and E. Lieb [**23**], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A readable account of this can be found in R. Baxter's book [**2**]. This represen-

Parallel with great discoveries in knot theory, closely related results came from integrable statistical mechanics, including quantum-group algebras and conformal field theory. Such interplay allowed the development of fusion categories.

To make this correspondence precise, we need to understand more about the algebraic structure underlying braiding and fusing.

The rules were systematized by Moore and Seiberg in order to understand chiral operators in 2D conformal field theory.

TAMING THE CONFORMAL ZOO

Gregory MOORE and Nathan SEIBERG [1]
Institute for Advanced Study, Princeton, NJ 08540, USA

Received 18 January 1989

All known rational conformal field theories may be obtained from (2+1)-dimensional Chern–Simons gauge theories by appropriate choice of gauge group. We conjecture that all rational field theories are classified by groups via (2+1)-dimensional Chern–Simons gauge theories.

1. Introduction

The problem of the classification of all conformal field theories is a useful problem to orient the research about the more interesting and more important problem of uncovering the meaning of conformal field theory, and, perhaps, string theory. An

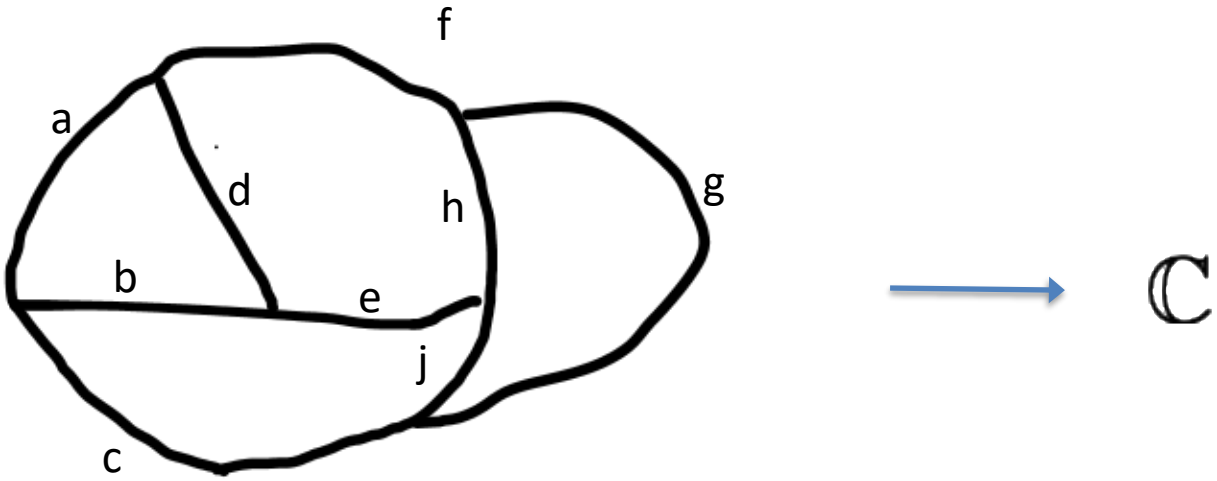
the structure uncovered in ref. [3] is neatly summarized by 3D general covariance. In ref. [6] the connection between two- and three-dimensional theories was established only for WZW models [8] based on a simply connected compact Lie group G. In this letter we show that all known RCFT's are equivalent to some CSGT thus organizing the entire zoo of known

Physics reviews: Moore and Seiberg, Kitaev, Bonderson...

A fusion category gives a set of rules to turn a fusion diagram into a number, an isotopy invariant

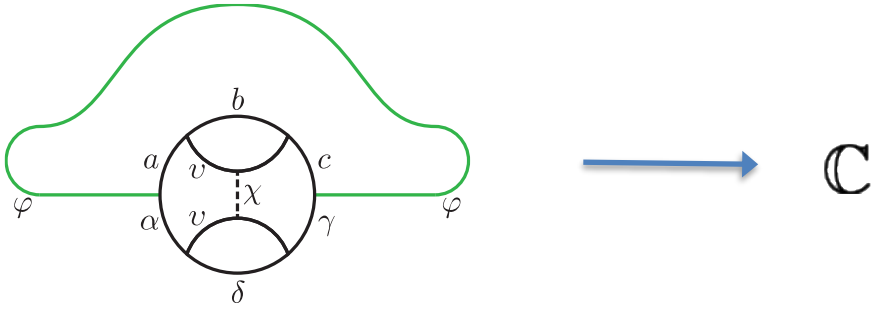
The Jones polynomial turns out to be the simplest example of an enormous class of topological invariants. Can't just use loops, but instead use a fusion diagram, i.e. a labelled trivalent graph.

The evaluation of such a graph is an isotopy invariant:



Isotopy invariance means that the evaluation is independent of any local deformations of the diagram.

Expressing diagram using a category



Category is a finite list of objects: a,b,c,… obeying fusion rules

$$a \otimes b = \sum_c N_{ab}^c \; c$$

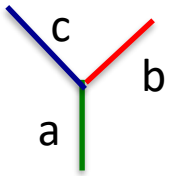
non-negative integer

Think tensor products of reps of Lie algebras, e.g. $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

The $SU(2)_k$ fusion rules are a truncated version of *SU*(2), with objects $0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$

Crucial connection:

Vertex for each $N_{bc}^a = 1$:



Manipulating the diagram

Data:

Non-negative integers N_{ab}^{c}

quantum dimensions d_a

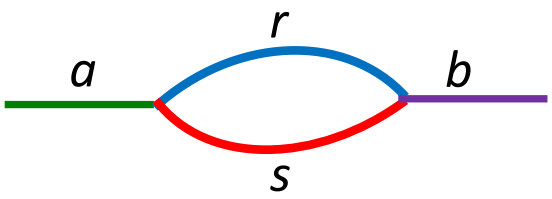
F symbols: $F_{tt'}\begin{bmatrix} r & s \\ a & b \end{bmatrix}$

Isolated loop:



$= d_r$

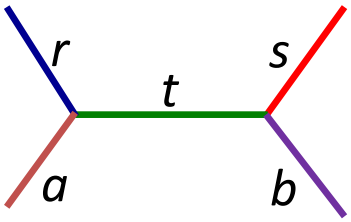
bubble removal:



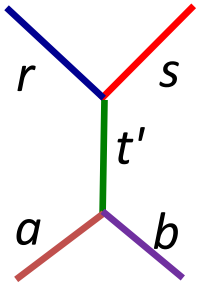
$= \delta_{ab} \sqrt{\frac{d_r d_s}{d_a}}$



F move:



$= \sum_{t' \in \mathcal{C}} F_{tt'} \begin{bmatrix} r & s \\ a & b \end{bmatrix}$

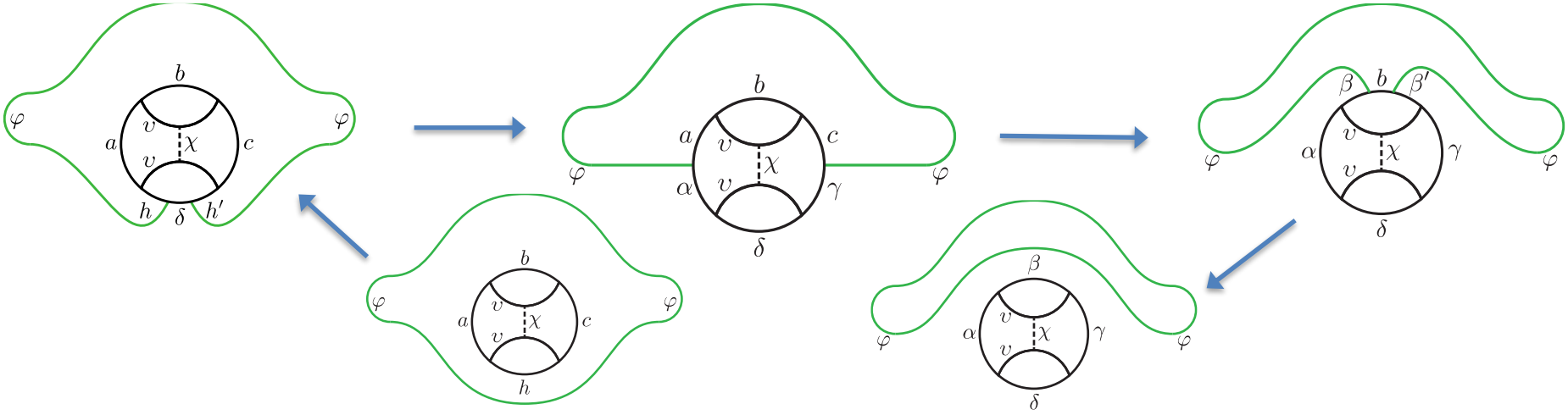


F moves preserve the evaluations. Note they result in a sum over diagrams.

Evaluating the diagram

F moves preserve the evaluations. Doing them and bubble removal repeatedly express a fusion diagram as a sum over closed loops, and hence a number.

Also very useful to relate different diagrams:

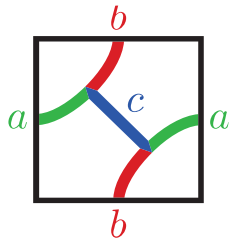


not writing sums and F symbols

Consistency guaranteed by the F symbols satisfying pentagon, hexagon equations.

Fusion categories allow one to construct topological defects both on the lattice and in the continuum

The partition function in the presence of topological defects is independent of local deformations of the defects. Just need to keep track of how they branch/fuse, and how they wrap around cycles. Schematically:



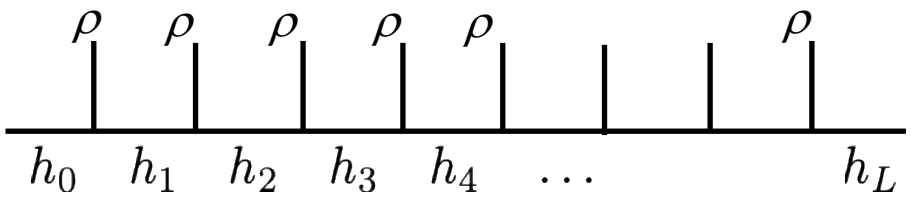
Working in a Hilbert-space formalism, the defect-line creation operators necessarily commute with the Hamiltonian/transfer matrix.

Topological defects generate non-invertible/categorical/ generalized ``dualities'' and symmetries.

The Hilbert space

To define model, first pick a category C, and a particular object $\rho \in C$.

Each basis element of Hilbert space is then given by a fusion tree:



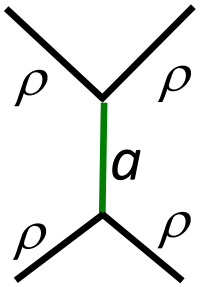
Degrees of freedom are "heights" $h_0, h_1, h_2, \ldots, h_L \in C$ obeying $N_{h_j \rho}^{h_{j+1}} = 1.$

(recall $a \otimes b = \sum_c N_{ab}^c \; c$)

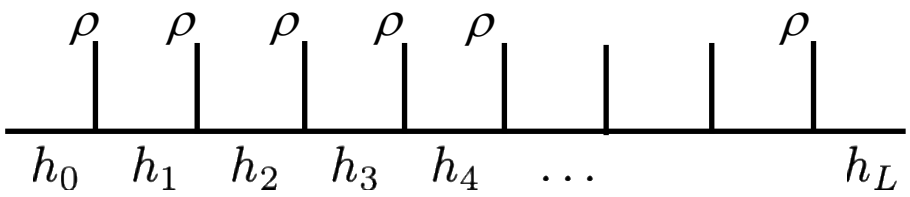
To get periodic boundary conditions, set $h_0 = h_L$.

Hamiltonian is any combination of projection operators

$P^{(a)} \propto$



Examples of Hilbert spaces



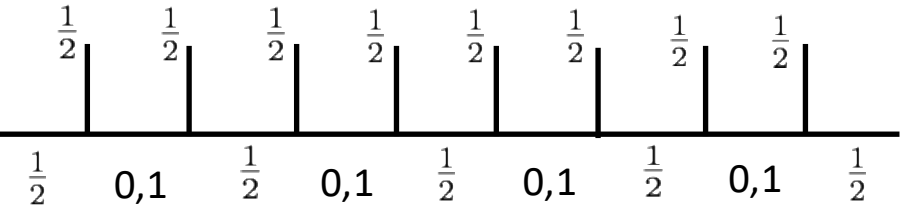
$$N_{h_j \rho}^{h_{j+1}} = 1$$

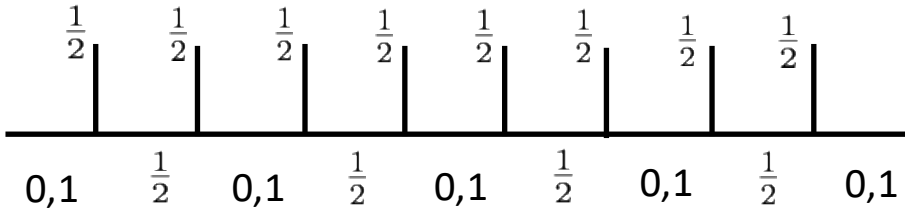
Ising/$SU(2)_2$

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \ , \qquad 1 \otimes \frac{1}{2} = \frac{1}{2} \ , \qquad a \otimes 0 = a$$

Take $\rho = \frac{1}{2}$.

Hilbert space then splits into two sectors:



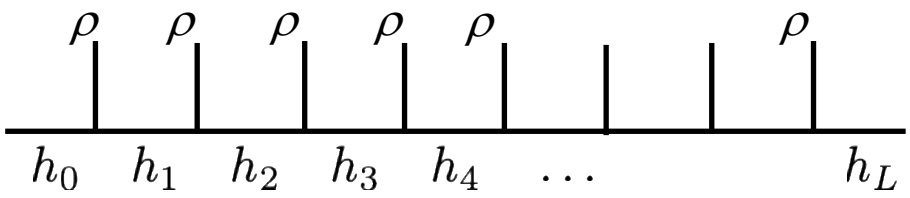


Within each sector, each such Hilbert space is simply a tensor product one!

2d classical model follows same rules for nearest neighbours:

$$\begin{array}{ccccccccc}
0 & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\
\frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\
1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\
\frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2}
\end{array}$$

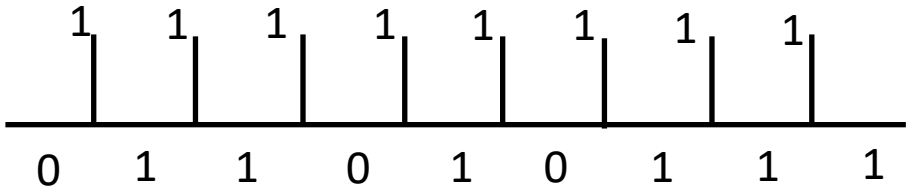
In general, not a tensor product



$$N_{h_j \rho}^{h_{j+1}} = 1$$

FIbonacci/hard squares/golden chain: $1 \otimes 1 = 0 \oplus 1$ with $\rho = 1$

e.g.



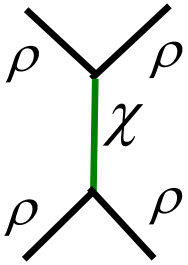
Cannot have two zeroes in a row, i.e. can't have $h_j = 0,\ h_{j+1} = 0$

If you think of 1 as being empty, and 0 as being a particle, get Rydberg blockade!

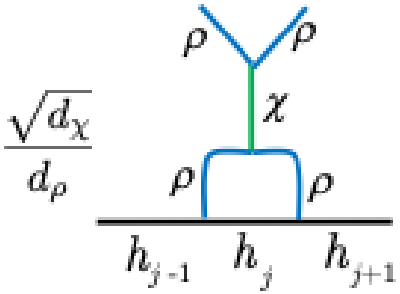
The Hamiltonian

Hamiltonian is any combination of projection operators

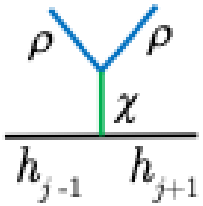
$$P^{(\chi)} = \frac{\sqrt{d_\chi}}{d_\rho}$$



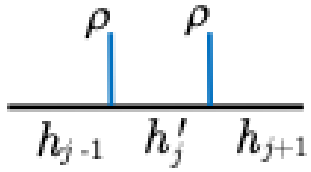
Each projector gives a linear operation on Hilbert space by gluing and using F moves:



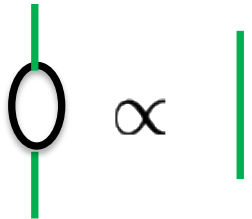
$$= F_{h_j\chi}\begin{bmatrix}\rho & \rho\\ h_{j-1} & h_{j+1}\end{bmatrix}$$



$$= \sum_{h'_j} F_{h_j\chi}\begin{bmatrix}\rho & \rho\\ h_{j-1} & h_{j+1}\end{bmatrix} F_{\chi h'_j}\begin{bmatrix}h_{j-1} & \rho\\ h_{j+1} & \rho\end{bmatrix}$$



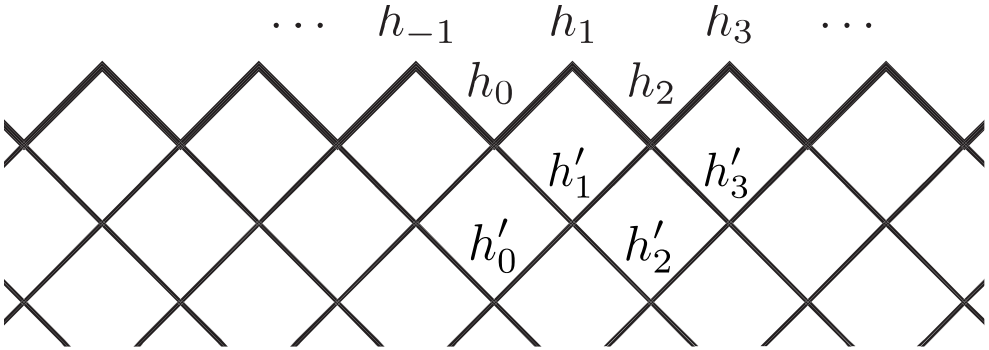
Use bubble removal to see that $P^{(\chi)}P^{(\chi')} = \delta_{\chi\chi'}P^{(\chi)}$

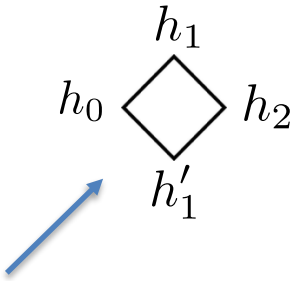


The beauty of the category!

Local statistical-mechanical models

Discrete degrees of freedom called ``heights'' living on the sites of a square lattice:





Boltzmann weight depends on interactions among the four heights ``round a face''

$$Z = \sum_{\text{heights}} \prod_{\text{faces}} \quad \begin{matrix} & h_j & \\ h_{j-1} & \diamond & h_{j+1} \\ & h'_j & \end{matrix}$$

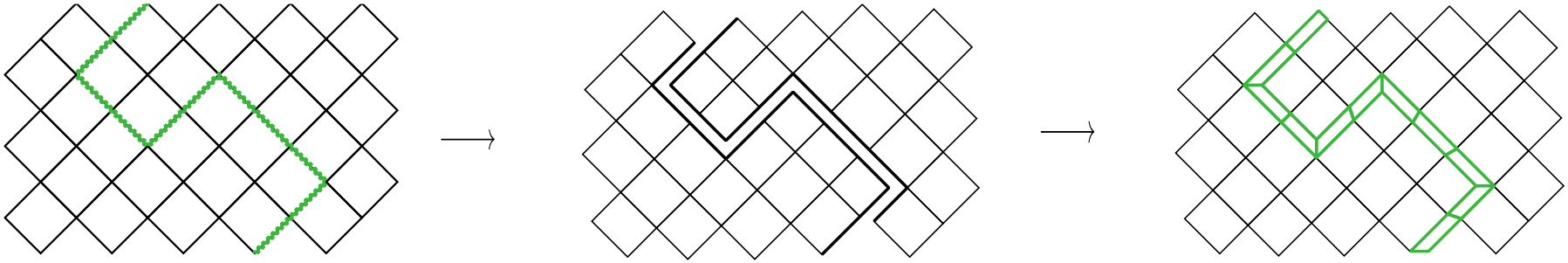
Like Hamiltonian, Boltzmann weights are built from projection operators $P^{(a)}$

For nearest-neighbor Ising, include degrees of freedom $a, b = 0, 1$ on half the sites:

$$\begin{matrix} & \text{a} & \\ \text{a} & \diamond & \text{b} \end{matrix} = e^{K_x \delta_{ab}}$$

$$\begin{matrix} & \text{a} & \\ & \diamond & \\ & \text{b} & \end{matrix} = e^{K_y \delta_{ab}}$$

Inserting a defect



The defects have a weight depending on the adjacent heights:

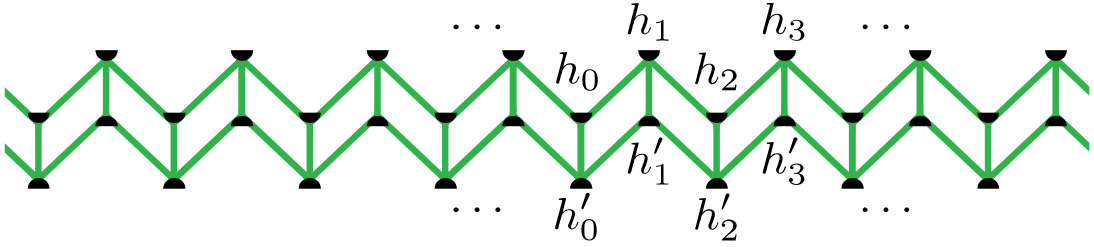


One type of defect for each object ϕ, with weight $\propto F_{\phi\rho}\begin{pmatrix} a & b' \\ b & a' \end{pmatrix}$

In the presence of the defect, the partition function is modified to

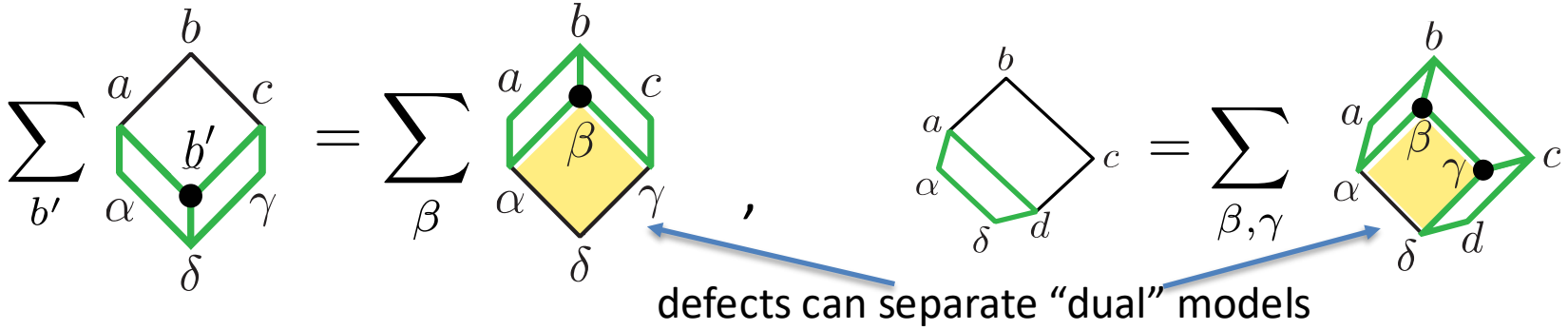
$$\mathcal{Z} = \sum_{\text{heights}} \prod_{\text{faces}} \diamond \times \prod_{\text{along defect}} \square$$

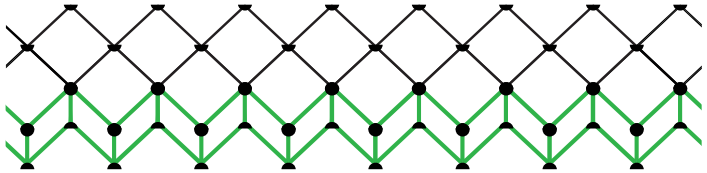
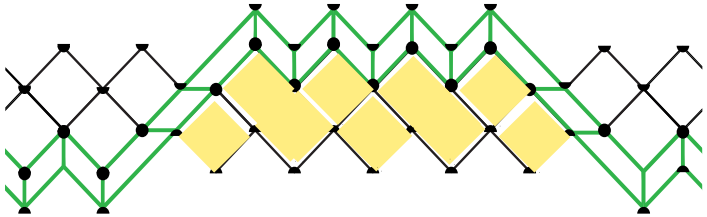
Defect-line creation operator: $\mathcal{D} =$

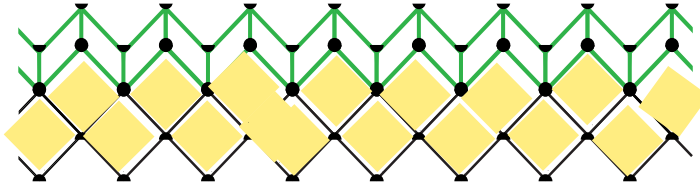


To make defects topological, they must satisfy defect commutation relations

For $\mathcal{Z}$ to be invariant under deformations of the defect's path:



$\mathcal{D}T_1 =$  = 

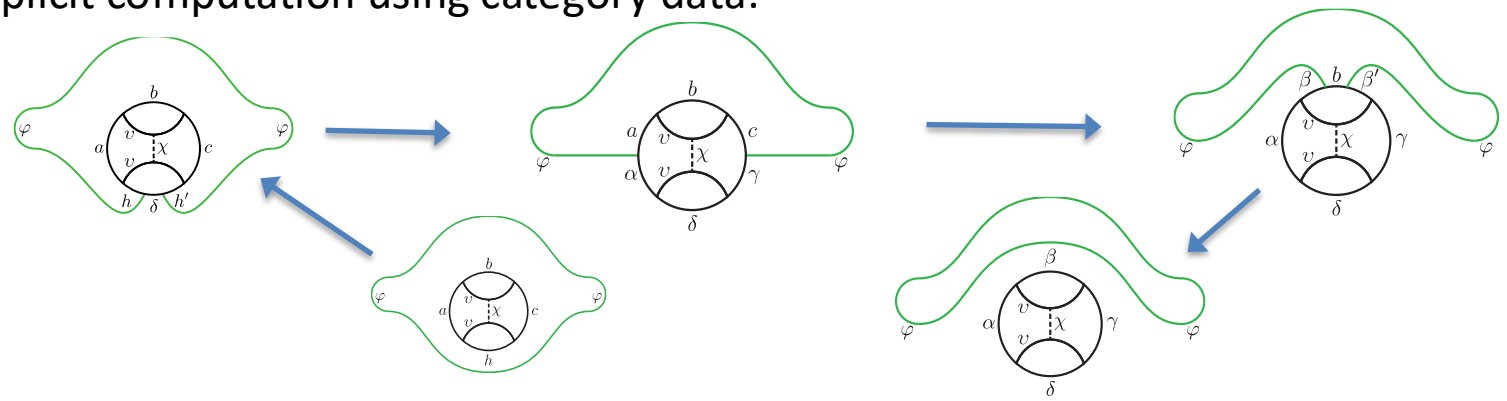
=  $= T_2\mathcal{D}$

Any lattice model built from category possesses solutions!

Multiple methods of proof

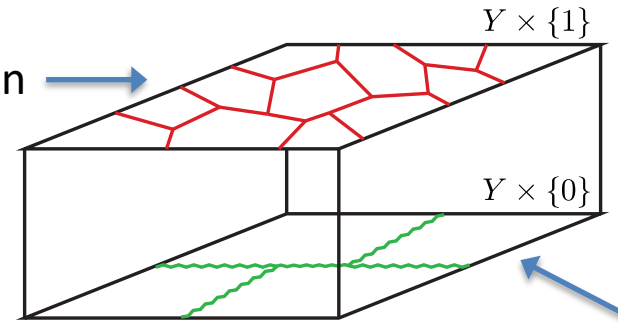
1) Defects are objects in the Drinfeld centre, so follows essentially by definition

2) Explicit computation using category data:



3) Using the Turaev-Viro-Barrett-Westbury ”state sum” approach to 3d TQFT (subsequently generalized and renamed ”SymTFT”)

Defect configuration



2d classical lattice model

Two types of topological defects in Ising

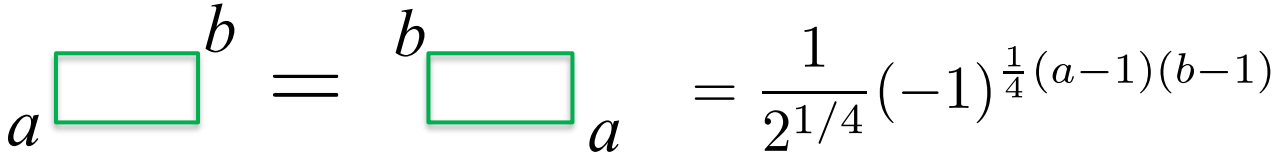
For Ising, fluctuating degrees of freedom on only half the sites:

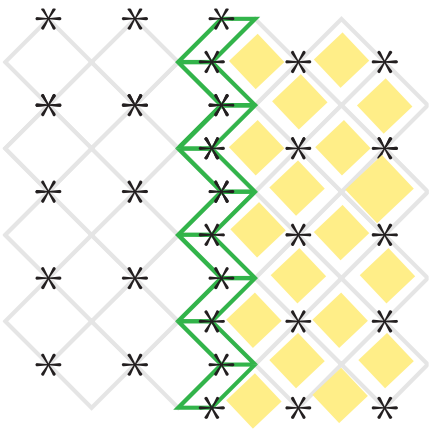


spin-flip defect:



duality defect:





Duality defect changes sublattice.
Coupings on one side of duality defect are dual to those on the other!

